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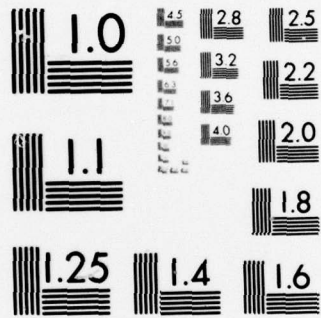   DEC 76   V CHVATAL, M R GAREY, D S JOHNSON   N00014-76-C-0330
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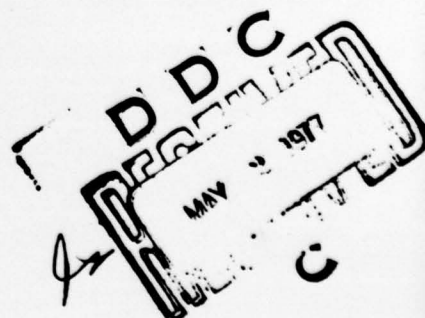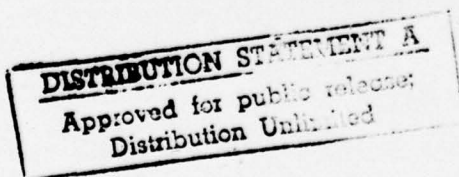
TWO RESULTS CONCERNING MULTICOLORING

by

V. Chvatal, M. R. Garey and D. S. Johnson

STAN-CS-76-582
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The m-chromatic number $X_m(G)$ of a graph $G = (V,E)$ is the least integer k such that there exists a mapping $f: V \to \{S \subseteq \{1,2,\ldots k\} : |S| = m\}$ having the property that $f(u) \cap f(v) = \emptyset$ whenever $\{u,v\} \in E$. This is a generalization of the standard notion of chromatic number and arises in connection with mobile telephone frequency assignments. Answering a question

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of Lovász, our first result shows that for any $m \geq 1$ and any $\varepsilon > 0$, there exists a graph G for which $X_{m+1}(G)/X_m(G) > 2-\varepsilon$. This shows that the known bound of 2 for all m and G is essentially best possible. Our second result shows that the least integer m_0 for which $X_{m_0}(G)/m_0 = \lim_{m \to \infty} X_m(G)/m$ can be asymptotically as large as $e^{\sqrt{(n\log n)/2}}$ for some n vertex graphs, though it can never exceed $e^{(n\log n)/2}$.

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ABSTRACT

The m-chromatic number $\chi_m(G)$ of a graph $G = (V,E)$ is the least integer k such that there exists a mapping $f:V \rightarrow \{S \subseteq \{1,2,\ldots,k\} : |S| = m\}$ having the property that $f(u) \cap f(v) = \phi$ whenever $\{u,v\} \in E$. This is a generalization of the standard notion of chromatic number and arises in connection with mobile telephone frequency assignments. Answering a question of Lovász, our first result shows that for any $m \geq 1$ and any $\epsilon > 0$, there exists a graph G for which $\chi_{m+1}(G)/\chi_m(G) > 2 - \epsilon$. This shows that the known bound of 2 for all m and G is essentially best possible. Our second result shows that the least integer m_0 for which $\chi_{m_0}(G)/m_0 = \lim_{m \to \infty} \chi_m(G)/m$ can be asymptotically as large as $e^{\sqrt{(n \log n)/2}}$ for some n vertex graphs, though it can never exceed $e^{(n \log n)/2}$.

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I. INTRODUCTION

The following generalization of the standard notion
of graph coloring has been of recent interest [1,3,4,6,7,8]. A
<u>multicoloring</u> of a graph $G = (V,E)$ is a function f defined on
V whose values are sets (of "colors") satisfying $f(u) \cap f(v) = \phi$
whenever $\{u,v\} \in E$. For positive integers k,m, a <u>(k,m)-coloring</u>
of $G = (V,E)$ is a multicoloring f of G such that $|f(v)| = m$
for each $v \in V$ and $|\underset{v \in V}{\cup} f(v)| = k$. The <u>m-chromatic number</u> $\chi_m(G)$
is the least integer k such that there exists a (k,m)-coloring
of G. (This last definition differs from that of [6,7] by a
factor of m.) Notice that for m = 1 these definitions corres-
pond to the usual graph coloring notions. The purpose of
this note is to resolve two questions about multicoloring con-
veyed to us by P. Erdös [2].

The first question deals with the relationship
between $\chi_m(G)$ and $\chi_{m+1}(G)$. It is not difficult to see that

$$\chi_{m+1}(G) \leq \chi_m(G) + \chi_1(G) \leq 2 \cdot \chi_m(G)$$

with equality possible in the right-hand inequality only
for m = 1. Lovász asked [2] whether, for each value of m,
there exist graphs G such that $\chi_{m+1}(G) > (2-\epsilon)\chi_m(G)$. We
shall answer this question in the affirmative.

The graphs we shall use are defined as follows: for posi-
tive integers n $\geq$ 2m, the graph G_m^n has vertex set consisting
of all m-element subsets of $\{1,2,\ldots,n\}$ and has an edge between
two such vertices exactly when their intersection is empty.
It is easy to see that $\chi_m(G_m^n) \leq n$ merely by considering the
multicoloring provided by the definition of G_m^n and, in fact,
it is proved in [7,8] that $\chi_m(G_m^n) = n$. Thus, to answer the
question of Lovász, it suffices to prove the following theorem:

Theorem 1. For each m $\geq$ 2, there exists a constant c such
that for all sufficiently large n

$$\chi_{m+1}(G_m^n) \geq 2n - c.$$

In order to prove Theorem 1, we require the following
lemma, which is an immediate consequence of a special case of
Theorem 3 in [5].

Lemma 1. For fixed m $\geq$ 2 and n sufficiently large, there
exists a constant a_0 such that the number of m-element subsets
of $\{1,2,\ldots,n\}$ which can be chosen so that no two are disjoint
but there is no element common to all is at most $a_0 n^{m-2}$.

Proof of Theorem 1. Fix m. We merely need to show that, for all sufficiently large n,

$$\chi_{m+1}(G_m^n) - \chi_{m+1}\left(G_m^{n-1}\right) \geq 2$$

and the result will follow by induction. So suppose we have a $(k,m+1)$-coloring of G_m^n such that $k = \chi_{m+1}\left(G_m^n\right)$, where n is any integer sufficiently large that the conclusion of Lemma 1 holds and such that $\binom{n-1}{m-1} > ma_0 n^{m-2}$, where a_0 is the constant of Lemma 1. We first claim that there must be at least n+1 colors which each appear on more than $a_0 n^{m-2}$ vertices.

Suppose there are n or fewer colors which each appear on more than $a_0 n^{m-2}$ vertices. By Lemma 1, each such color can appear on at most $\binom{n-1}{m-1} > a_0 n^{m-2}$ vertices since they must all share a common element. Thus, since each of the $\binom{n}{m}$ vertices receives exactly m+1 colors, we must have

$$(m+1)\ \binom{n}{m} \leq n\ \binom{n-1}{m-1} + (k-n)a_0 n^{m-2}$$

$$\leq m\ \binom{n}{m} + (k-n)a_0 n^{m-2}$$

or, rewriting,

$$\binom{n}{m} \leq (k-n)a_0 n^{m-2}$$

Since

$$k = \chi_{m+1}\left(G_m^n\right) \leq \chi_m\left(G_m^n\right) + \chi\left(G_m^n\right)$$

$$\leq 2\chi_m\left(G_m^n\right) = 2n$$

it follows that we must have

$$\binom{n}{m} \le a_0 n^{m-1} \cdot$$

However this is a contradiction, since n was chosen sufficiently large that $\binom{n}{m} = \frac{n}{m} \binom{n-1}{m-1} > \frac{n}{m} m a_0 n^{m-2} = a_0 n^{m-1}$, and the claim follows.

Thus there are at least n+1 colors which each appear on more than $a_0 n^{m-2}$ vertices. The set of vertices on which any color i appears must form a collection of pairwise-intersecting m-element subsets of $\{1,2,\ldots,n\}$, by definition of G_m^n. Thus, by Lemma 1, whenever color i appears on more than $a_0 n^{m-2}$ vertices, all those vertices must contain some common element e_i. Since there are more than n such colors, we must have $e_i = e_j$ for some i and j. If we delete from G_m^n all the vertices containing $e_i = e_j$, we obtain a copy of G_m^{n-1} and a (k-2,m+1)-coloring of it, since colors i and j have disappeared. Therefore

$$\chi_{m+1} \left(G_m^{n-1}\right) \le k-2 = \chi_{m+1} \left(G_m^n\right) - 2$$

and the theorem is proved. ∎

The second question involves what we call the ultimate multichromatic number $\chi^*(G)$ defined by

$$\chi^*(G) = \inf_m \chi_m(G)/m.$$

It is proved in [1,7] that the value of $\chi^*(G)$ is always achieved
for some finite m. One easy way to see this is to formulate
the problem of determining $\chi^*(G)$ as a linear programming
problem (as done in [4]): Let $v_1, v_2, \ldots, v_n$ be an ordering
of the vertices of G and let $S_1, S_2, \ldots, S_\ell$ be an ordering of
the independent sets of G. Define x_{ij} to be 1 whenever $v_i \in S_j$
and 0 otherwise. Then the value of $\chi^*(G)$ is given by

$$\chi^*(G) = \min \sum_{j=1}^{\ell} r_j$$

subject to: $r_j \geq 0$, $1 \leq j \leq \ell$;

$$\sum_{j=1}^{\ell} x_{ij} r_j = 1, \quad 1 \leq i \leq n.$$

One can show easily, using Hadamard's Theorem, that no basis
matrix for this problem can have determinant exceeding $n^{n/2}$
and this is an upper bound on the value of m required.

This upper bound however seems ridiculously large.
Erdös asked [2] (as did the authors, independently) whether
$\chi^*(G)$ could always be achieved for an m not exceeding the
number of vertices of G. We answer this in the negative,
constructing graphs for which extremely large values of m are
necessary to achieve $\chi^*(G)$.

Let C_p denote the graph which is a cycle on p vertices. The join $G_1 + G_2$ of two graphs G_1 and G_2, having disjoint vertex sets, consists of all edges and vertices in the two given graphs along with all edges joining a vertex from G_1 to a vertex from G_2. We use the following two lemmas in our construction:

Lemma 2. [4,7] For all integers $p \geq 1$,

$$\chi^*(C_{2p+1}) = 2 + (1/p).$$

Lemma 3. [7] For all graphs G_1 and G_2,

$$\chi^*(G_1 + G_2) = \chi^*(G_1) + \chi^*(G_2).$$

Let p_i denote the ith prime and define the graph $G(i)$ to be $C_{2p_1+1} + C_{2p_2+1} + \cdots + C_{2p_i+1}$. The number of vertices n of $G(i)$ is given by

$$n = i + 2 \sum_{j=1}^{i} p_j.$$

Applying Lemmas 2 and 3, we obtain

$$\chi^*(G(i)) = 2i + \sum_{j=1}^{i} (1/p_j).$$

Since $\chi_m(G(i))$ must always be an integer, it follows that the least value of m for which $\chi^*(G(i)) = \chi_m(G(i))/m$ can be no less that $\prod_{j=1}^{i} p_j$ (and in fact that value of m will work). Using the Prime Number Theorem and expressing this lower bound in

terms of n, we obtain the asymptotic lower bound of

$$e^{\sqrt{(n\log n)/2}} .$$

Thus, though this is still quite far from the upper bound of

$$n^{n/2} = e^{(n\log n)/2}$$

we see that extremely large values of m <u>can</u> be required in order to achieve $\chi^*(G)$.

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